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(Residential Autonomous College affiliated to University of Calcutta)
B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2019
THIRD YEAR [BATCH 2017-20]
MATHEMATICS [Honours]
Paper : V

Date : 14/12/2019
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Answer any five questions from Question Nos. 1 to 8 : [5×10]

1. a) Let H be a finite subgroup of a group G. If G contains no other subgroup with $o(H)$ elements, then prove that H is normal in G. Deduce that $\{i,(1,2)(3,4),(1,3)(2,4),(1,4),(2,3)\}$ is normal in A_4.
   b) Let H be a subgroup of G. If $[G:H]=2$ prove that H is a normal subgroup of G. Is the converse true? Justify your answer.
   c) Let H be a normal subgroup of G and $[G:H]=3$. Prove that $a^3 \in H \ \forall a \in G$. [(3+2)+3+2]

2. a) Suppose $G=(\{z\in\mathbb{C} : |z|=1\},\cdot)$ where '$\cdot$' is the usual multiplication of complex numbers. Show that G is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$.
   b) Find the centre of the dihedral group D_4.
   c) Show that a cyclic group of order 4 cannot be expressed as an internal direct product of two subgroups of order 2. [3+4+3]

3 a) (i) Prove that $\mathbb{Z}\times\mathbb{Z}$ is not cyclic.
   (ii) Find the number of elements of order 5 in $\mathbb{Z}_{15}\times\mathbb{Z}_5$.
   b) Let G be a group of order p^2 where p is prime. Prove that G is commutative.
   c) Find the class equation for S_3. [(2+2)+2+4]

4. a) State and prove Sylow's first theorem.
   b) Show that a group of order 96 has a normal subgroup of order 16 or 32. [5+5]

5. a) (i) Does there exist a ring epimorphism from $\mathbb{R}$ onto $\mathbb{Z}$? Justify.
   (ii) Show that any non-zero ideal in the ring $\mathbb{Z}+i\mathbb{Z}=\{a+ib : a,b\in\mathbb{Z}\}$ contains a positive integer.
   b) Let R be a ring with identity having the property that $a^2=a,\ \forall a\in R$. Prove that every non-zero prime ideal in R is a maximal ideal.
   c) Show that the field $\mathbb{Q}$ has no proper subfield. [(2+3)+3+2]

6. a) Show that [2] is a prime element in $\mathbb{Z}_{10}$, but not an irreducible element in $\mathbb{Z}_{10}$.
   b) In an integral domain, prove that any associate of an irreducible element is irreducible.
   c) Prove that an integral domain D with ACCP (ascending chain condition for principal ideals) is a factorization domain. [3+3+4]

7. a) Prove that in a unique factorization domain, every irreducible element is prime.

   b) Is $x^2+[1]$ irreducible in $\mathbb{Z}_2[x]$? Justify.

   c) Find all prime elements in $\mathbb{Z}_9$. [5+3+2]

8. a) Let R be a commutative ring with identity and M be an ideal of R. Show that M is a maximal ideal if and only if R/M is a field.

   b) Let F be a finite field. Construct a polynomial over F which has no root in F.

   c) What are the units of the ring $\mathbb{Z}_5[x]$? [5+2+3]

<u>Group – B</u>

***Answer <u>any three</u> questions from <u>Question Nos. 9 to 13</u>* :** [3×5]

9. Let $f(x,y)=\dfrac{\sin x+\sin 2y}{\tan 2x+\tan y}$. Prove that the repeated limits exist but the double limit does not exist. Cite an example of a function for which the double limit exists but repeated limits do not exist. [3+2]

10. Let a differentiable function f(x,y) transforms into g(u,v) under the substitutions

$$x=\frac{1}{2}(u+v),\ y=\sqrt{uv}.$$

Show that $\dfrac{\partial^2 g}{\partial u\partial v}=\dfrac{1}{4}\left(\dfrac{\partial^2 f}{\partial x^2}+2\dfrac{x}{y}\dfrac{\partial^2 f}{\partial x\partial y}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{1}{y}\dfrac{\partial f}{\partial y}\right)$.

11. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=|xy|^{\frac{1}{2}}$

    (a) Does the directional derivative exists at the origin in any arbitrary direction $\beta,(\|\beta\|=1)$?

    (b) Check whether f is differentiable at the origin. [2+3]

12. Let S be an open set in $\mathbb{R}^2$ and f: S$\to\mathbb{R}$ be a mapping. If (a,b)$\in$ S and both f_x, f_y are differentiable at (a,b), prove that $f_{xy}(a,b)=f_{yx}(a,b)$.

13. Define

$$f(x,y)=\begin{cases} x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y} & ;x\neq 0, y\neq 0\\ 0 & ;x=0 \text{ or } y=0\end{cases}$$

Show that $f_{xy}(0,0)\,f_{yx}(0,0)=-1$.

***Answer <u>any one</u> questions from <u>Question Nos. 14 to 15</u>* :** [1×15]

14. (a) If H is a homogeneous function in x,y,z of degree n having continuous partial derivatives of

second order, prove that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right)=0$$

where $u=\left(x^2+y^2+z^2\right)^{-\frac{n+1}{2}}$

(b) Prove that the quadratic forms

$ax^2+2hxy+by^2$ and $lx^2+2mxy+ny^2$ are independent unless $\frac{a}{l}=\frac{h}{m}=\frac{b}{n}$. [8+7]

15. (a) Use the method of Lagrange multipliers to show that the lengths of the semi-axes of ellipse $ax^2+2hxy+by^2=1$ are the square roots of the roots of the equation

$$\begin{vmatrix} a\lambda-1 & h\lambda \\ h\lambda & b\lambda-1 \end{vmatrix}=0.$$

(b) Derive the approximate formula $\frac{\cos x}{\cos y}\simeq 1-\frac{1}{2}\left(x^2-y^2\right)$ for sufficiently small values of $|x|,|y|$.

(c) Let f(x,y) and g(x,y,u) be such that $f_x g_y - f_y g_x = 0$ where u=f(x,y). Show that f(x,y) and g(x,y,u) are functionally dependent. [5+5+5]

Answer <u>any four</u> questions from <u>Question Nos. 18 to 23</u> **:** [4×5]

16. Show that a function of bounded variation is bounded but the converse is not true. [3+2]

17. Let $f:[a,b]\to\mathbb{R}$ be a Lipschitz function of order 1. Show that f is a function of bounded variation on [a,b]. Is the converse true? If not, give an example showing that the converse is not true. [3+2]

18. Let the function $h:[0,4]\to\mathbb{R}$ be defined by $h(x)=\frac{x^3}{3}-\frac{3x^2}{2}+2x,\ 0\le x\le 4$. Show that h is a function of bounded variation on [0,4]. Also calculate $V_h[0,4]$. [3+2]

19. Show that the second Mean value theorem(Bonnet's form) is applicable to $\int_a^b \frac{\sin x}{x}dx$ where $0<a<b<\infty$. Also prove that $\left|\int_a^b \frac{\sin x}{x}dx\right|\le\frac{2}{a}$. [2+3]

20. Let $f:[a,b]\to\mathbb{R}$ be a bounded function on [a,b]. Let f possesses finite number of discontinuities on [a,b]. Then show that f is R-integrable on [a,b].

21. Show that $\frac{\pi}{6}<\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-k^2x^2\right)}}<\frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{k^2}{4}}}$ for $k^2<1$.

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